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THE EXTENDED LENGTH BIASED TWO PARAMETERS MIRRA DISTRIBUTION WITH AN APPLICATION TO ENGINEERING DATA

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Abstract. In this article, we introduce an extension of the two parameters Mirra distribution (MD) called length biased two parameters Mirra distribution (MD), which offers more flexibility in modeling some life time data. Some statistical properties of the suggested model are explicitly derived. These properties include the moments, moment generating function, the coefficient of variation, coefficient of skewness and coefficient of kurtosis, the distributions of order statistics, reliability analysis and the maximum likelihood estimators of the model parameters. Also, we obtain the Rényi entropy, stochastic ordering, mean and median deviations, Lorenz and Bonferroni curves and the Gini index. An application of the model to a real data set consists of the failure times of 84 Aircraft Windshield is evaluated and compared with some other well-known existing distributions.

Keywords: Mirra distribution, Maximum likelihood estimators, Reliability analysis, Rényi entropy, Gini index, Stochastic ordering, Lorenz curve, Bonferroni curve.
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1 Introduction

Sen et al. (2017) suggested the two parameters Mirra distribution (MD) with probability density function (pdf) given by

$$f_{MD}(x;\alpha,\theta)=\frac{\theta^3}{\alpha+\theta^2}\left(1+\frac{\alpha}{2}x^2\right)\mathrm{e}^{-\theta x},\ x>0,\ \ \alpha>0,\theta>0, \tag{1}$$

and cumulative distribution function (cdf) defined as

$$F_{MD}(x;\alpha,\theta)=1-\frac{\theta^2\left(1+\frac{\alpha}{\theta^2}+\frac{\alpha}{\theta}x+\frac{\alpha}{2}x^2\right)}{\alpha+\theta^2}\mathrm{e}^{-\theta x}. \tag{2}$$

The rth mean of the MD is

$$E\left(X^r\right)=\frac{\theta^{-r}\Gamma(r+1)\left(2\theta^2+\alpha(r+1)(r+2)\right)}{2\left(\alpha+\theta^2\right)},\ r=1,2,3,.... \tag{3}$$

and for $r=1$ we obtain the mean of the distribution as $\mu=\frac{3\alpha+\theta^2}{\alpha\theta+\theta^3}$. The mode of the MD is $M(X)=\frac{1+\sqrt{1-2\theta^2/\alpha}}{\theta}$, $if\,\theta^2<\alpha/2$, and zero otherwise.

In 1934, Fisher introduced the concept of weighted distributions for modeling ascertainment bias. Let X be a random variable with pdf $f(x)$, then the corresponding weighted distribution function is defined by

$$f^w(x)=\frac{w(x)f(x)}{E(w(x))}, \tag{4}$$

where $w(x)$ is a non-negative weight function such that $E(w(x))$ exists. Rao (1965) defined the size-biased distributions which is a special case of the weighted distributions when the weighted function is given by $w(x) = x^a$ with pdf

$$f_a(x) = \frac{x^a f(x)}{E_a(X)}. \tag{5}$$

For $a = 1, 2$ we can get the length-biased and size-biased distributions, respectively. Recently, the idea of weighted distribution attracts the researchers to introduce several weighted distributions for fitting real data. For instance, see Gupta and Kundu (2009) for a new class of weighted exponential distributions; Ahmed et al. (2013) for size biased Gamma distribution; Mudasir and Ahmad (2015) for length biased Nakagami distribution; Reyad et al. (2017) for length-biased weighted Erlang distribution; Al-Kadim and Fadhil (2018) for double weighted Lomax distribution; Al-Omari et al. (2019) for size-biased Ishita distribution; Al-Omari and Alsmairan (2019) for length-biased Suja distribution; Alsmairan and Al-Omari (2020) for weighted Suja distribution; Sherina and Oluyede (2014) proposed weighted inverse Weibull distribution; Modi and Gill (2017) proposed exponentiated generalized Lindley distribution. Beghriche and Beghriche (2019) proposed a size biased gamma Lindley distribution. Zeghdoudi and Nedjar (2016) proposed gamma Lindley distribution.

The paper is organized as follows: In Section 2, we define the length-biased Mirra distribution and shapes of the pdf and cdf are presented. Some distributional properties are given in Section 3. The applicability of the suggested model is accomplished in Section 4. Finally, the paper is concluded in Section 5.

2 The extended length biased two parameters Mirra distribution

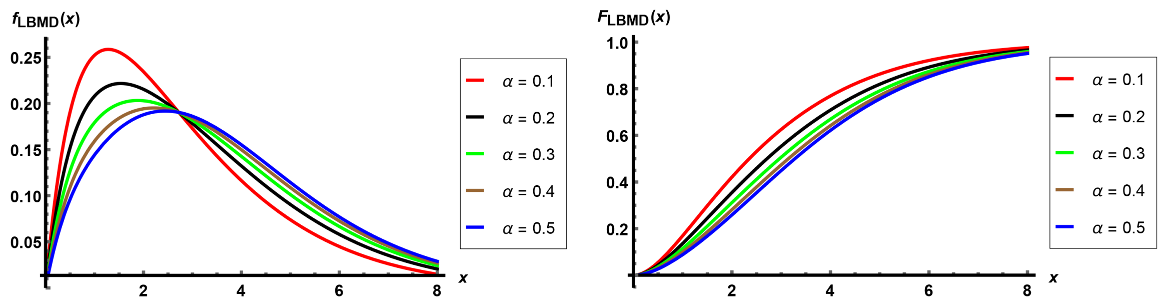
The pdf of the length biased two parameters Mirra distribution is defined as

$$f_{LBMD}(x;\alpha,\theta) = \frac{\theta^4}{3\alpha+\theta^2} x \left(1 + \frac{x^2\alpha}{2}\right) \mathrm{e}^{-x\theta},\ 0 < x < \infty,\ \alpha > 0. \tag{6}$$

It is clear that $f_{LBMD}(x;\alpha,\theta)$ satisfies the conditions of being a pdf, that is $f(x) \geq 0$ for all x and $\int\limits_0^\infty f_{LBMD}(x;\theta,\alpha)dx = 1$. The corresponding cumulative distribution function of the LBMD is

$$F_{LBMD}(x;\alpha,\theta) = 1 - \frac{\alpha\left[6\theta x + (\theta x)^3 + 3(\theta x)^2 + 6\right] + 2\theta^2(\theta x + 1)}{2\left(3\alpha + \theta^2\right)} \mathrm{e}^{-x\theta},\ x > 0,\ \alpha > 0, \theta > 0. \tag{7}$$

In Figure 1 we plot the pdf and cdf of the LBMD for some values of θ and α.



(a) (b)

Figure 1: Plots of pdf and cdf of the LBMD for $\alpha = 0.1, 0.2, 0.3, 0.4, 0.5$ and $\theta = 0.9$.

Figure 1 indicates that the pdf of the LBMD is asymmetric and positively skewed depending on the distribution parameters values.

3 Distributional Properties

This section is devoted in finding important distributional properties of the LBMD

3.1 Raw and central moments

The rth moment of the LBMD is

$$E\left(X_{LBMD}^{r}\right)=\frac{\left[(2+r)\left(3+r\right)\alpha+2\theta^{2}\right]\Gamma\left[2+r\right]}{2\theta^{r}\left(3\alpha+\theta^{2}\right)},\ \alpha,\theta>0,\ r=1,2,3,... \tag{8}$$

For $r=1,2,3,4$ in Equation (8), the first four moments of the LBMD distribution, respectively, are given as

$$E\left(X_{LBMD}^{1}\right)=\frac{2\left(6\alpha+\theta^{2}\right)}{\theta\left(3\alpha+\theta^{2}\right)}=\mu, E\left(X_{LBMD}^{2}\right)=\frac{6\left(10\alpha+\theta^{2}\right)}{\theta^{2}\left(3\alpha+\theta^{2}\right)},$$

$$E\left(X_{LBMD}^{3}\right)=\frac{12\left(30\alpha+2\theta^{2}\right)}{\theta^{3}\left(3\alpha+\theta^{2}\right)}, E\left(X_{LBMD}^{4}\right)=\frac{120\left(21\alpha+\theta^{2}\right)}{\theta^{4}\left(3\alpha+\theta^{2}\right)},\ \theta>0.$$

Hence, the variance of the distribution is

$$\mathrm{Var}(X_{LBMD})=E\left(X_{LBMD}^{2}\right)-\left(E\left(X_{LBMD}^{1}\right)\right)^{2}=\frac{2\left(18\alpha^{2}+15\alpha\theta^{2}+\theta^{4}\right)}{\left(3\alpha\theta+\theta^{3}\right)^{2}}. \tag{9}$$

3.2 Moment generating function

The moment generating function, $M_X(t)$ of the LBMD is

$$M_{LBMD}(t)=\frac{\theta^{4}}{3\alpha+\theta^{2}}\frac{3\alpha+\left(\theta-t\right)^{2}}{\left(\theta-t\right)^{4}},\theta>0,\ \alpha>0. \tag{10}$$

The characteristic function (CF) of LBMD is

$$\varphi_{LBMD}(t)=\frac{\theta^{4}}{3\alpha+\theta^{2}}\frac{3\alpha+\left(\theta-it\right)^{2}}{\left(\theta-it\right)^{4}},t\in R,\ i=\sqrt{-1}. \tag{11}$$

3.3 Coefficient of skewness

$$Sk_{LBMD}\frac{\sqrt{2}\left(54\alpha^{3}+54\alpha^{2}\theta^{2}+27\alpha\theta^{4}+\theta^{6}\right)}{\left(18\alpha^{2}+15\alpha\theta^{2}+\theta^{4}\right)^{3/2}} \tag{12}$$

3.4 Coefficient of kurtosis

$$Ku_{LBMD}=\frac{6\left(243\alpha^{4}+378\alpha^{3}\theta^{2}+180\alpha^{2}\theta^{4}+36\alpha\theta^{6}+\theta^{8}\right)}{\left(18\alpha^{2}+15\alpha\theta^{2}+\theta^{4}\right)^{2}}. \tag{13}$$

3.5 Coefficient of variation

The coefficient of variation is

$$Cv_{LBMD} = \frac{\sqrt{18\alpha^2 + 15\alpha\theta^2 + \theta^4}}{\sqrt{2}\,(6\alpha + \theta^2)}. \tag{14}$$

Tables (1) and (2) contain the mean, standard deviation (SD), the coefficient of variation, the coefficient of skewness and coefficient of kurtosis for some selected values of the distribution parameters.

Table 1: The mean, SD, CV, SK and CK for $\alpha = 0.6, 2$ and some selected values of θ.

	θ	Mean	SD	CV	SK	CK
$\alpha = 0.6$	1	3.285710	2.05039	0.624030	1.02109	4.43037
↑	2	1.310340	0.932311	0.711500	1.30484	5.32169
↑	3	0.777778	0.566558	0.728431	1.44070	5.98669
↑	4	0.550562	0.400456	0.727359	1.47819	6.24368
↑	5	0.426866	0.308865	0.723565	1.48091	6.30378
↑	6	0.349206	0.251476	0.720135	1.47388	6.29235
↑	7	0.295838	0.212256	0.717473	1.46522	6.26015
↑	8	0.256839	0.183760	0.715469	1.45730	6.22534
↑	9	0.227053	0.162106	0.713956	1.45064	6.19373
↑	10	0.203536	0.145081	0.712800	1.44519	6.16670
↑	11	0.184483	0.131334	0.711903	1.44075	6.14408
↑	12	0.168724	0.119996	0.711196	1.43713	6.12526
↑	13	0.155467	0.110480	0.710631	1.43417	6.10960
↑	14	0.144157	0.102377	0.710173	1.43171	6.09652
↑	15	0.134392	0.095391	0.709798	1.42967	6.08552
$\alpha = 2$	1	3.714290	2.050390	0.552027	0.95107	4.35762
↑	2	1.600000	1.019800	0.637377	1.04847	4.48891
↑	3	0.933333	0.646357	0.692526	1.21576	4.97872
↑	4	0.636364	0.456813	0.717848	1.34281	5.48696
↑	5	0.477419	0.347070	0.726972	1.41842	5.86046
↑	6	0.380952	0.277664	0.728869	1.45774	6.09213
↑	7	0.316883	0.230658	0.727895	1.47547	6.21961
↑	8	0.271429	0.197044	0.725953	1.48139	6.28077
↑	9	0.237548	0.171945	0.723832	1.48120	6.30280
↑	10	0.211321	0.152539	0.721835	1.47806	6.30299
↑	11	0.190408	0.137105	0.720060	1.47366	6.29168
↑	12	0.173333	0.124544	0.718521	1.46890	6.27486
↑	13	0.159121	0.114122	0.717201	1.46420	6.25591
↑	14	0.147100	0.105334	0.716072	1.45980	6.23668
↑	15	0.136797	0.097824	0.715106	1.45575	6.21816

Based on Table 1, the distribution mean, standard deviation and coefficient of variation values are decreasing as theta values are increasing for fixed values of alpha. But the coefficients of skewness and kurtosis values are decreasing up to some values then start increasing. Table 2 shows that the distribution kurtosis average value is about 4.4 for all cases in the table.

Table 2: The mean, SD, CV, SK and CK for $\theta = 0.6, 2$ and some selected values of α.

	α	Mean	SD	CV	SK	CK
$\theta = 0.6$	1	6.30952	3.40276	0.539306	0.953903	4.37941
$\uparrow$	2	6.47799	3.37490	0.520980	0.967782	4.42614
$\uparrow$	3	6.53846	3.36279	0.514309	0.976002	4.44738
$\uparrow$	4	6.56958	3.35611	0.510857	0.980973	4.45921
$\uparrow$	5	6.58854	3.35190	0.508746	0.984262	4.46672
$\uparrow$	6	6.60131	3.34900	0.507323	0.986589	4.47190
$\uparrow$	7	6.61049	3.34688	0.506298	0.988320	4.47568
$\uparrow$	8	6.61741	3.34526	0.505525	0.989657	4.47857
$\uparrow$	9	6.62281	3.34399	0.504921	0.990720	4.48085
$\uparrow$	10	6.62714	3.34297	0.504436	0.991586	4.48269
$\uparrow$	11	6.63070	3.34212	0.504038	0.992304	4.48421
$\uparrow$	12	6.63366	3.34141	0.503705	0.992909	4.48548
$\uparrow$	13	6.63618	3.34081	0.503423	0.993426	4.48656
$\uparrow$	14	6.63834	3.34029	0.503181	0.993873	4.48750
$\uparrow$	15	6.64021	3.33984	0.502971	0.994264	4.48831
$\theta = 2$	1	1.42857	0.97938	0.685565	1.188640	4.88569
$\uparrow$	2	1.60000	1.01980	0.637377	1.048470	4.48891
$\uparrow$	3	1.69231	1.02916	0.608140	0.994214	4.38165
$\uparrow$	4	1.75000	1.03078	0.589015	0.970143	4.34948
$\uparrow$	5	1.78947	1.03002	0.575600	0.958847	4.34236
$\uparrow$	6	1.81818	1.02852	0.565685	0.953627	4.34464
$\uparrow$	7	1.84000	1.02684	0.558065	0.951518	4.35051
$\uparrow$	8	1.85714	1.02519	0.552027	0.951074	4.35762
$\uparrow$	9	1.87097	1.02365	0.547125	0.951545	4.36494
$\uparrow$	10	1.88235	1.02224	0.543067	0.952517	4.37203
$\uparrow$	11	1.89189	1.02096	0.539652	0.953755	4.37870
$\uparrow$	12	1.90000	1.01980	0.536739	0.955123	4.38489
$\uparrow$	13	1.90698	1.01875	0.534224	0.956540	4.39061
$\uparrow$	14	1.91304	1.01780	0.532032	0.957956	4.39586
$\uparrow$	15	1.91837	1.01693	0.530103	0.959345	4.40069

Table 3: The MLE and MSE of the estimated LBMD parameters for some selected values.

	$\alpha=2$		$\theta=3$		$\alpha=0.5$		$\theta=0.9$	
N	$\hat{\alpha}$	$MSE(\hat{\alpha})$	$\hat{\theta}$	$MSE(\hat{\theta})$	$\hat{\alpha}$	$MSE(\hat{\alpha})$	$\hat{\theta}$	$MSE(\hat{\theta})$
50	2.2228	1.6028	3.0240	0.0486	0.6569	0.4241	0.9048	0.0040
100	2.1218	0.7492	3.0093	0.0244	0.5637	0.1189	0.9036	0.0021
150	2.0730	0.4433	3.0077	0.0156	0.5423	0.0392	0.8999	0.0014
200	2.0586	0.3484	3.0056	0.0124	0.5159	0.0256	0.9039	0.0010
300	2.0525	0.2381	3.0014	0.0087	0.5179	0.0148	0.9008	0.0006
500	2.0081	0.1238	3.0049	0.0048	0.5100	0.0078	0.9005	0.0004
600	2.0194	0.1043	3.0021	0.0042	0.5075	0.0068	0.9002	0.0003
700	2.0108	0.0879	3.0019	0.0035	0.5078	0.0056	0.9007	0.0003
800	2.0062	0.0773	3.0031	0.0032	0.5026	0.0044	0.9007	0.0002
900	2.0094	0.0711	3.0023	0.0028	0.5077	0.0047	0.8998	0.0002
1000	2.0116	0.0607	3.0010	0.0024	0.5046	0.0043	0.9004	0.0002
1100	2.0265	0.0510	2.9975	0.0020	0.5061	0.0036	0.8997	0.0002

3.6 Parameters estimation

Let $x_1, x_2, ..., x_n$ be the sample observations of the sample $X_1, X_2, ..., X_n$ chosen from the LBMD. The likelihood function for the LBMD is defined as

$$L_{LBMD}(\alpha,\theta|x_1,x_2,...,x_n)=\prod_{i=1}^{n}\frac{\theta^4}{3\alpha+\theta^2}x_i\left(1+\frac{x_i^2\alpha}{2}\right)\mathrm{e}^{-\theta x_i}$$

The log-likelihood function is given by

$$\begin{aligned}\log\left[L_{LBMD}(\alpha,\theta|x_1,x_2,...,x_n)\right]&=n\log\left(\frac{\theta^4}{3\alpha+\theta^2}\right)+\log\left(\prod_{i=1}^{n}x_i\right)+\log\prod_{i=1}^{n}\left(1+\frac{x_i^2\alpha}{2}\right)-\theta\sum_{i=1}^{n}x_i\\&=n\log\left(\frac{\theta^4}{3\alpha+\theta^2}\right)+\sum_{i=1}^{n}\log\,(x_i)+\sum_{i=1}^{n}\log\,\left(1+\frac{x_i^2\alpha}{2}\right)-\theta\sum_{i=1}^{2}x_i.\end{aligned} \tag{15}$$

Differentiating (15) partially with respect to α and θ, respectively and setting the result equals to zero, we have the log likelihood equations as

$$\frac{\partial\log\left[L_{LBMD}(\alpha,\theta|x_1,x_2,...,x_n)\right]}{\partial\alpha}=-\frac{3n}{3\alpha+\theta^2}+\sum_{i=1}^{n}\frac{x_i^2}{2\left(1+\frac{\alpha x_i^2}{2}\right)} \tag{16}$$

$$\frac{\partial\log\left[L_{LBMD}(\alpha,\theta|x_1,x_2,...,x_n)\right]}{\partial\theta}=\frac{n}{\theta}\left(-\frac{2\theta^2+4\left(3\alpha+\theta^2\right)}{3\alpha+\theta^2}\right)-n\,\bar{x}. \tag{17}$$

By solving this nonlinear system of Equations (16) and (17) numerically such as Newton-Raphson we get the ML estimators for α and θ. Table 3 lists the estimated distribution parameters using the MLE for some parameters selection. It turns out that the MSE values are decreasing as the sample size are increasing for all cases given in Table 3.

3.7 Reliability analysis

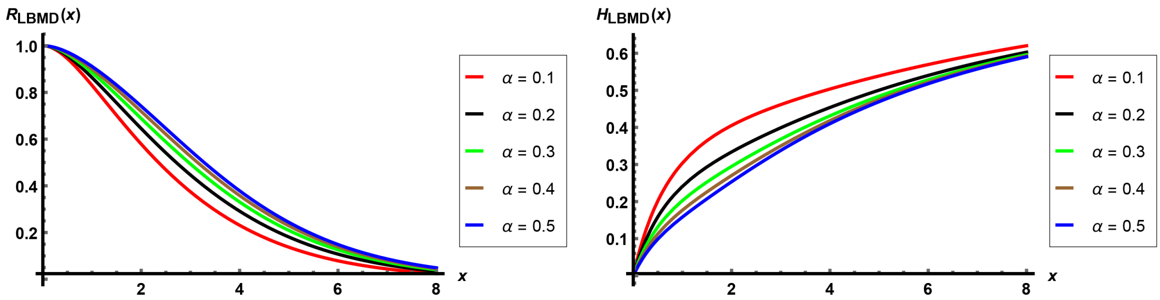
This section introduces the reliability analysis of the suggested LBMD. The reliability and hazard rate functions of the LBMD distribution are, respectively given by

$$R_{LBMD}(x,\alpha,\theta)=1-F_{LBMD}(x,\alpha,\theta)=\frac{\alpha\left[6\theta x+(\theta x)^3+3(\theta x)^2+6\right]+2\theta^2(\theta x+1)}{2\left(3\alpha+\theta^2\right)}\mathrm{e}^{-x\theta}, \tag{18}$$

and

$$H_{LBMD}(x,\alpha,\theta)=\frac{f_{LBMD}(x,\alpha,\theta)}{1-F_{LBMD}(x,\alpha,\theta)}=\frac{\theta^4 x\left(2+x^2\alpha\right).}{\alpha\left[6\theta x+(\theta x)^3+3(\theta x)^2+6\right]+2\theta^2(\theta x+1)} \tag{19}$$

Figure (2) explores the plots of the $R_{LBMD}(x;\theta,\alpha)$ and $H_{LBMD}(x;\theta,\alpha)$ for $\alpha=0.1$, 0.2, 0.3, 0.4, 0.5 and $\theta=0.9$.



(a) (b)

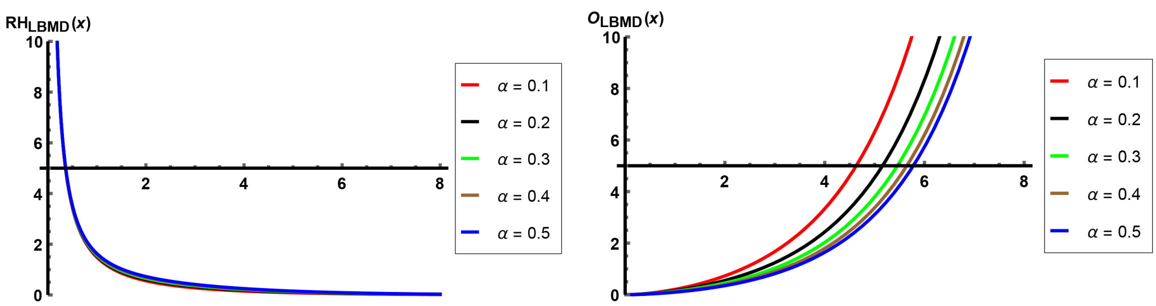
Figure 2: Plots of the reliability and hazard functions of the LBMD for $\alpha=0.1,0.2,0.3,0.4,0.5$ and $\theta=0.9$.

The odds and the reversed hazard rate functions of the LBMD respectively, are defined as

$$H_{LBMD}(x,\alpha,\theta)=\frac{f_{LBMD}(x,\alpha,\theta)}{1-F_{LBMD}(x,\alpha,\theta)}=\frac{\theta^4 x\left(2+x^2\alpha\right).}{\alpha\left[6\theta x+(\theta x)^3+3(\theta x)^2+6\right]+2\theta^2(\theta x+1)} \tag{20}$$

$$\begin{aligned}O_{LBMD}(x;\alpha,\theta)=&\frac{F_{LBMD}(x;\alpha,\theta)}{1-F_{LBMD}(x;\alpha,\theta)}\\=&\frac{2\left(3\alpha+\theta^2\right)-\left\{\alpha\left[6\theta x+(\theta x)^3+3(\theta x)^2+6\right]+2\theta^2(\theta x+1)\right\}\mathrm{e}^{-x\theta}}{\alpha\left[6\theta x+(\theta x)^3+3(\theta x)^2+6\right]+2\theta^2(\theta x+1)\mathrm{e}^{-x\theta}}.\end{aligned} \tag{21}$$

The possible odds and the reversed hazard rate functions shapes of LBMD distribution are displayed in Figure (3) for some selected parameters values.



(a) (b)

Figure 3: The reverse hazard and odds functions shapes of the LBMD for $\alpha=0.1,0.2,0.3,0.4,0.5$ and $\theta=0.9$.

3.8 Order statistics

Let $X_{(1:m)},X_{(2:m)},...,X_{(m:m)}$ be the order statistics of a random sample of size m, $X_1,X_2,...,X_m$ selected from $f_{LBMD}(x;\alpha,\theta)$ with $F_{LBMD}(x;\alpha,\theta)$, respectively. The pdf of the ith order statistics say $X_{(i:m)}$ is

$$\begin{aligned} f_{(j:m)}(x;\alpha,\theta) &= \frac{m!}{(j-1)!(m-j)!}[F(x;\alpha,\theta)]^{j-1}[1-F(x;\alpha,\theta)]^{m-j}f(x;\alpha,\theta) \\ &= \frac{\theta^4 x m! e^{-\theta x(m-j+1)}\left(\alpha x^2+2\right)}{2^m(3\alpha+\theta^2)^m\Gamma(j)\Gamma(m+1-j)}\left[\alpha\left\{\theta x(\theta x(\theta x+3)+6)+6\right\}+2\theta^2(\theta x+1)\right]^{m-j} \\ &\times\left(2\left(3\alpha+\theta^2\right)-\left[\alpha\left\{\theta x(\theta x(\theta x+3)+6)+6\right\}+2\theta^2(\theta x+1)\right]e^{-\theta x}\right)^{j-1} \end{aligned} \tag{22}$$

Hence, the pdf of minimum and maximum order statistics say $X_{(1:m)}$ and $X_{(m:m)}$, are respectively, defined by

$$f_{(1:m)}(x;\theta,\alpha) = \frac{\theta^4 m x e^{-\theta m x}\left(\alpha x^2+2\right)\left[\alpha(\theta x(\theta x(\theta x+3)+6)+6)+2\theta^2(\theta x+1)\right]^{m-1}}{2^m(3\alpha+\theta^2)^m} \tag{23}$$

$$f_{(m:m)}(x;\theta,\alpha) = \frac{\theta^4 m x e^{-\theta x}\left(\alpha x^2+2\right)\left[2\left(3\alpha+\theta^2\right)-\left(\begin{array}{c}\alpha(\theta x(\theta x(\theta x+3)+6)+6)\\+2\theta^2(\theta x+1)\end{array}\right)e^{-\theta x}\right]^{m-1}}{2^m(3\alpha+\theta^2)^m}. \tag{24}$$

3.9 Stochastic ordering

The stochastic ordering can be used to compare two positive continuous distributions. A random variable X is smaller than a random variable Y in

- mean residual life order $X \underset{MRLO}{\leq} Y$ if $m_X(x) \leq m_Y(x)$ for all x.
- likelihood ratio order $X \underset{LRO}{\leq} Y$ if $\frac{f_X(x)}{f_Y(x)}$ decreases for all x.
- likelihood ratio order $X \underset{HRO}{\leq} Y$ if $h_X(x) \geq h_Y(x)$ for all x.
- likelihood ratio order $X \underset{SO}{\leq} Y$ if $F_X(x) \geq F_Y(x)$ for all x.

It is shown by Shaked and Shanthikumar (1994) that

$$\begin{array}{c} X \underset{LRO}{\leq} Y \Rightarrow X \underset{HRO}{\leq} Y \Rightarrow X \underset{MRLO}{\leq} Y \\ \Downarrow \\ X \underset{SO}{\leq} Y \end{array}$$

Theorem 1. *Let $X_{LBMD} \sim f_X(x;\alpha,\theta)$ and $Y_{LBMD} \sim f_Y(x;\delta,\lambda)$, then if $\lambda < \theta$, we have $X \underset{LRO}{\leq} Y$, and hence $X \underset{HRO}{\leq} Y$ and $X \underset{MRLO}{\leq} Y$, and $X \underset{SO}{\leq} Y$.*

Proof. Let $X_{LBMD} \sim f_X(x;\alpha,\theta)$ and $Y_{LBMD} \sim f_Y(x;\delta,\lambda)$. Hence,

$$f_{LBMD}(x;\alpha,\theta) = \frac{\theta^4}{3\alpha+\theta^2}x\left(1+\frac{x^2\alpha}{2}\right)\mathrm{e}^{-x\theta},$$

and

$$f_{LBMD}(x;\delta,\lambda) = \frac{\lambda^4}{3\delta+\lambda^2}x\left(1+\frac{x^2\delta}{2}\right)\mathrm{e}^{-\lambda x}$$

Therefore,

$$\frac{f_X(x;\alpha,\theta)}{f_Y(x;\delta,\lambda)} = \frac{\frac{\theta^4}{3\alpha+\theta^2} x\left(1+\frac{x^2\alpha}{2}\right)\mathrm{e}^{-x\theta}}{\frac{\lambda^4}{3\delta+\lambda^2} x\left(1+\frac{x^2\delta}{2}\right)\mathrm{e}^{-\lambda x}} = \frac{\theta^4\left(3\delta+\lambda^2\right)}{\lambda^4\left(3\alpha+\theta^2\right)}\frac{2+x^2\alpha}{2+x^2\delta}e^{-(\theta-\lambda)x}$$

and its logarithm is

$$\begin{aligned}\ln\frac{f_X(x;\alpha,\theta)}{f_Y(x;\delta,\lambda)} &= \ln\left[\frac{\theta^4\left(3\delta+\lambda^2\right)}{\lambda^4\left(3\alpha+\theta^2\right)}\frac{2+x^2\alpha}{2+x^2\delta}e^{-(\theta-\lambda)x}\right]\\ &= \ln\left(\frac{\theta^4}{\lambda^4}\right)+\ln\left(\frac{3\delta+\lambda^2}{3\alpha+\theta^2}\right)+\ln\left(\frac{2+x^2\alpha}{2+x^2\delta}\right)-(\theta-\lambda)\,x.\end{aligned}$$

The derivative of the last equation with respect to x is

$$\begin{aligned}\frac{\partial}{\partial x}\ln\frac{f_X(x;\alpha,\theta)}{f_Y(x;\delta,\lambda)} &= -\theta+\lambda+\frac{\left(\delta x^2+2\right)\left(\frac{2\alpha x}{\delta x^2+2}-\frac{2\delta x\left(\alpha x^2+2\right)}{\left(\delta x^2+2\right)^2}\right)}{\alpha x^2+2}\\ &= -\theta+\lambda+\frac{4x(\alpha-\delta)}{\left(\alpha x^2+2\right)\left(\delta x^2+2\right)}.\end{aligned}$$

Now, if $\lambda<\theta$, then $\frac{\partial}{\partial x}\ln\left(\frac{f_X(x;\alpha,\theta)}{f_Y(x;\delta,\lambda)}\right)<0$, which implies that $X \underset{LRO}{\leq} Y$, and hence $X \underset{HRO}{\leq} Y$, $X \underset{MRLO}{\leq} Y$ and $X \underset{SO}{\leq} Y$. The proof is completed. □

3.10 Mean and median deviations

The mean deviations can be used as a measure of spread in a population. The mean deviations about the mean($D(\mu)$) and about the median ($D(m)$) are given by the following:

$$D(\mu)=\int_0^\infty |x-\mu|f(x)dx = 2\mu F(\mu)-2\int_0^\mu xf(x)dx$$

and

$$D(m)=\int_0^\infty |x-m|f(x)dx = \mu-2\int_0^M xf(x)dx$$

where $\mu=E(X)$ and $m=median(X)$. For the LBMD we have

$$D(\mu)=\frac{8\left(4\alpha+\theta^2\right)\left(6\alpha+\theta^2\right)^3}{\theta(3\alpha+\theta^2)^4}e^{-\frac{6\alpha}{3\alpha+\theta^2}-2} \tag{25}$$

and

$$D(m)=e^{-m\theta}\frac{\alpha\left[\theta m\left(\theta m[\theta m(\theta m+4)+12]+24\right)-12e^{\theta m}+24\right]+2\theta^2\left[\theta m(\theta m+2)-e^{\theta m}+2\right]}{3\alpha\theta+\theta^3}. \tag{26}$$

3.11 Lorenz and Bonferroni curves and Gini Index

Let X be a non-negative random variable with a continuous and twice differentiable cumulative distribution function $F(x)$. The Lorenz curve is commonly used in economics in order to represent the distribution of income and the inequality of wealth distribution. Lorenz (1905) defined this curve as

$$L(t) = \frac{\int_0^t xf(x)dx}{\int_0^\infty xf(x)dx}.$$

For the LBMD it is defined as

$$L_{LBMD}(t) = 1 + \frac{e^{-\theta t}\left(-\alpha\left[\theta t(\theta t(\theta t(\theta t+4)+12)+24)+24\right]-2\theta^2\left(\theta t(\theta t+2)+2\right)\right)}{4\left(6\alpha+\theta^2\right)} \tag{27}$$

The Bonferroni curve of X is defined as

$$B(p) = \frac{1}{p\mu}\int_0^p F^{-1}(t)dt,\ p \in [0,1]$$

For the LBMD it is given by

$$B_{LBMD}(t) = \frac{\alpha\left[-\theta t(\theta t(\theta t(\theta t+4)+12)+24)+24e^{\theta t}-24\right]-2\theta^2\left[\theta t(\theta t+2)-2e^{\theta t}+2\right]}{4p\left(6\alpha+\theta^2\right)}e^{-\theta t} \tag{28}$$

The Gini index, G, is given by

$$G = 1 - \frac{1}{\mu}\int_0^\infty (1-F(x))^2 dx = \frac{1}{\mu}\int_0^\infty F(x)(1-F(x))dx.$$

The Gini index for the LBMD is

$$G_{LBMD} = \frac{3}{64}\left(-\frac{21\alpha}{3\alpha+\theta^2}+\frac{29\alpha}{6\alpha+\theta^2}+8\right). \tag{29}$$

Table (4) reveals the Gini index values of the LBMD for $\theta = 1, 2, ..., 14$ and $\alpha = 1, 2, 3, 4, 5$. According to Table (4), the Gini index values are all less than 1 and are running from 0.2 to 0.4 for all values of the distribution parameters given in the table.

3.12 Stress-strength reliability

Let the random variables X and Y are independent and observed from the pdf $f(x)$. The stress-strength reliability explain the life of a component that has a random strength Y that is subjected to a random stress X, and it is defined as

$$R = P(Y < X) = \int_0^\infty P(Y < X|X = x)\, f(x)dx = \int_0^\infty f(x;\alpha)F(x;\delta)dx$$

Theorem 2. *Let X and Y be two independent random variables follow the LBMD. Then, the stress-strength reliability is given by*

$$R_{LBMD} = \frac{\beta^4}{\left(3\alpha+\theta^2\right)\left(\beta^2+3\delta\right)(\beta+\theta)^7}\left[\begin{array}{l} 3\alpha\left(\begin{array}{l}\beta^5+3\beta^3\delta+35\theta^3\left(\beta^2+3\delta\right)\\ +21\beta\theta^2\left(\beta^2+3\delta\right)\\ +7\beta^2\theta\left(\beta^2+3\delta\right)+30\beta\theta^4+10\theta^5\end{array}\right)\\ +\theta^2(\beta+\theta)^2\left(\begin{array}{l}\beta^3+5\beta^2\theta+3\beta\delta\\ +7\beta\theta^2+15\delta\theta+3\theta^3\end{array}\right)\end{array}\right] \tag{30}$$

Table 4: Gini index values of the LBMD for some vales of θ and α.

θ	$\alpha = 1$	$\alpha = 2$	$\alpha = 3$	$\alpha = 4$	$\alpha = 5$
1	0.370313	0.302885	0.294326	0.289615	0.286637
2	0.383594	0.348047	0.333206	0.323103	0.315862
3	0.384981	0.373214	0.361979	0.352273	0.344201
4	0.383695	0.382610	0.376820	0.370313	0.363988
5	0.382126	0.384972	0.382983	0.379550	0.375533
6	0.380786	0.384766	0.384896	0.383594	0.381476
7	0.379727	0.383774	0.384952	0.384937	0.384132
8	0.378906	0.382648	0.384279	0.384981	0.385005
9	0.378267	0.381605	0.383381	0.384447	0.384964
10	0.377765	0.380701	0.382467	0.383695	0.384485
11	0.377366	0.379940	0.381623	0.382895	0.383822
12	0.377045	0.379303	0.380872	0.382126	0.383107
13	0.376783	0.378771	0.380218	0.381419	0.382406
14	0.376567	0.378325	0.379651	0.380786	0.381748
α	$\theta = 1$	$\theta = 2$	$\theta = 3$	$\theta = 4$	$\theta = 5$
1	0.323103	0.370313	0.383594	0.384981	0.383695
2	0.302885	0.348047	0.373214	0.382610	0.384972
3	0.294326	0.333206	0.361979	0.376820	0.382983
4	0.289615	0.323103	0.352273	0.370313	0.379550
5	0.286637	0.315862	0.344201	0.363988	0.375533
6	0.284584	0.310440	0.337500	0.358138	0.371354
7	0.283083	0.306236	0.331893	0.352829	0.367228
8	0.281939	0.302885	0.327153	0.348047	0.363258
9	0.281037	0.300151	0.323103	0.343745	0.359493
10	0.280308	0.297880	0.319607	0.339871	0.355949
11	0.279707	0.295964	0.316563	0.336373	0.352628
12	0.279202	0.294326	0.313889	0.333206	0.349523
13	0.278773	0.292909	0.311523	0.330328	0.346620
14	0.278403	0.291672	0.309417	0.327705	0.343908

3.13 Rényi entropy

The Rényi entropy is defined as

$$RE(\eta) = \frac{1}{1-\eta} \log \left(\int_{0}^{\infty} f(x)^{\eta} dx \right), \eta > 0, \eta \neq 1$$

Theorem 3. *Let $X \sim f_{LBMD}(x;\alpha,\theta)$, then the and Rényi entropy of X is defined as*

$$RE(\eta) = \left[\frac{\log}{1-\eta} \left(\frac{\left(\frac{\theta^4}{3\alpha+\theta^2}\right)^{\eta}}{\Gamma(-\eta)} \left(\begin{array}{l} -2^{\eta/2}\eta\theta\alpha^{-\frac{\eta}{2}-1}\Gamma\left(-\frac{3\eta}{2}-1\right)\Gamma\left(\frac{\eta}{2}+1\right) {}_1F_2\left(\frac{\eta}{2}+1; \frac{3}{2}, \frac{3\eta}{2}+2; -\frac{\eta^2\theta^2}{2\alpha}\right) \\ +2^{\frac{\eta-1}{2}}\alpha^{-\frac{\eta}{2}-\frac{1}{2}}\Gamma\left(\frac{\eta+1}{2}\right)\Gamma\left(-\frac{3\eta}{2}-\frac{1}{2}\right) {}_1F_2\left(\frac{\eta}{2}+\frac{1}{2}; \frac{1}{2}, \frac{3\eta}{2}+\frac{3}{2}; -\frac{\eta^2\theta^2}{2\alpha}\right) \\ +2^{-\eta}\alpha^{\eta}(\eta\theta)^{-3\eta-1}\Gamma(-\eta)\Gamma(3\eta+1) {}_1F_2\left(-\eta; \frac{1}{2}-\frac{3\eta}{2}, -\frac{1}{2}(3\eta); -\frac{\eta^2\theta^2}{2\alpha}\right) \end{array} \right) \right) \right] \tag{31}$$

Table 5, revealed that the Rényi entropy values are all positive and depend on the parameters values, where it is increasing for some values and decreasing for other values.

The harmonic mean of length biased LBMD distribution is given in the following theorem.

Table 5: Renyi entropy values for some selected LBMD distribution parameters.

α	$\eta = 0.7$	$\eta = 0.7$	$\eta = 0.4$	$\eta = 0.4$	θ	$\alpha = 0.4$	$\alpha = 0.4$	$\alpha = 0.9$	$\alpha = 0.9$
	$\theta = 4$	$\theta = 2$	$\theta = 4$	$\theta = 2$		$\eta = 0.7$	$\eta = 0.7$	$\eta = 0.3$	$\eta = 0.7$
2	0.56527	1.41525	0.78902	1.60543	0.1	4.43615	4.43404	4.75079	4.74915
3	0.61847	1.43983	0.83124	1.62579	0.4	3.06927	3.06318	3.37040	3.37075
4	0.65410	1.45067	0.85902	1.63560	0.7	2.49337	2.50968	2.79204	2.80859
5	0.67923	1.45589	0.87854	1.64091	1	2.08990	2.14043	2.40173	2.43843
6	0.69762	1.45846	0.89289	1.64399	1.3	1.77149	1.85061	2.10178	2.15534
7	0.71146	1.45965	0.90379	1.64584	1.6	1.51102	1.60764	1.85842	1.92343
8	0.72210	1.46008	0.91228	1.64696	1.9	1.29389	1.39821	1.65489	1.72644
9	0.73043	1.46008	0.91903	1.64763	2.2	1.10999	1.21524	1.48102	1.55542
10	0.73705	1.45984	0.92448	1.64802	2.5	0.95188	1.05396	1.33001	1.40472
12	0.74668	1.45897	0.93264	1.64827	3.1	0.69234	0.78268	1.07863	1.14966
13	0.75021	1.45845	0.93573	1.64825	3.4	0.58368	0.66738	0.97211	1.04021
14	0.75311	1.45792	0.93834	1.64816	3.7	0.48571	0.56289	0.87546	0.94036
15	0.75552	1.45738	0.94056	1.64804	4	0.39661	0.46762	0.78711	0.84870
16	0.75752	1.45684	0.94245	1.64788	4.3	0.31496	0.38025	0.70583	0.76412
17	0.75920	1.45632	0.94408	1.64771	4.6	0.23966	0.29970	0.63061	0.68570
18	0.76059	1.45581	0.94549	1.64752	4.9	0.16979	0.22506	0.56066	0.61266
19	0.76176	1.45532	0.94670	1.64733	5.2	0.10465	0.15559	0.49529	0.54437
20	0.76275	1.45484	0.94777	1.64713	5.5	0.04363	0.90661	0.43397	0.48030

Theorem 4. *The harmonic mean of LBMD distribution is given by*

$$H_{LBMD}(x) = \frac{\theta}{3\alpha + \theta^2}\left(\alpha + \theta^2\right) \tag{32}$$

Table 6 shows that the LBMD harmonic mean vales are increasing as the parameter theta increasing for fixed $\alpha = 3,\ 7,\ 0.5,\ 0.1$. But for increasing alfa values the distribution harmonic mean values are decreasing for all $\theta = 3,\ 7,\ 0.5,\ 0.1$.

4 Applications of real data

In this section, the LBMD distribution is compared with some competitive models to illustrate its performance in real data modeling. We compare the LBMD with several competitive models such as

• Darna distribution (DD) with pdf

$$f(x) = \frac{\theta}{2\alpha^2 + \theta^2}\left(2\alpha + \frac{\theta^4 x^2}{2\alpha^3}\right) e^{-\frac{\theta x}{\alpha}};\ x > 0,\ \alpha > 0,\ \theta > 0.$$

• Janardan distribution (JD) with pdf

$$f(x) = \frac{\theta^2}{\alpha(\theta + \alpha^2)}(1 + \alpha x)e^{-\frac{\theta}{\alpha}x}; x > 0,\ \theta > 0,\ \alpha > 0.$$

• Two parameters Sujatha distribution (TSPD) with pdf

$$f(x) = \frac{\theta^3}{\alpha\theta^2 + \theta + 2}\left(\alpha + x + x^2\right) e^{-\theta x}; x > 0,\ \theta > 0.$$

Table 6: Harmonic mean values of LBMD for some selected parameters.

θ	$\alpha=3$	$\alpha=7$	$\alpha=0.5$	$\alpha=0.1$	α	$\theta=3$	$\theta=7$	$\theta=0.5$	$\theta=0.1$
1	0.40000	0.36364	0.60000	0.60000	1	2.50000	6.73077	0.19231	0.03356
2	1.07692	0.88000	1.63636	1.63636	2	2.20000	6.49091	0.18000	0.03345
3	2.00000	1.60000	2.71429	2.71429	3	2.00000	6.27586	0.17568	0.03341
4	3.04000	2.48649	3.77143	3.77143	4	1.85714	6.08197	0.17347	0.03339
5	4.11765	3.47826	4.81132	4.81132	5	1.75000	5.90625	0.17213	0.03338
6	5.20000	4.52632	5.84000	5.84000	6	1.66667	5.74627	0.17123	0.03337
7	6.27586	5.60000	6.86139	6.86139	7	1.60000	5.60000	0.17059	0.03337
8	7.34247	6.68235	7.87786	7.87786	8	1.54545	5.46575	0.17010	0.03336
9	8.40000	7.76471	8.89091	8.89091	9	1.50000	5.34211	0.16973	0.03336
10	9.44954	8.84298	9.90148	9.90148	10	1.46154	5.22785	0.16942	0.03336
11	10.4923	9.91549	10.9102	10.9102	11	1.42857	5.12195	0.16917	0.03335
12	11.5294	10.9818	11.9175	11.9175	12	1.40000	5.02353	0.16897	0.03335
13	12.5618	12.0421	12.9238	12.9238	13	1.37500	4.93182	0.16879	0.03335
14	13.5902	13.0968	13.9291	13.9291	14	1.35294	4.84615	0.16864	0.03335
15	14.6154	14.1463	14.9338	14.9338	15	1.33333	4.76596	0.16851	0.03335
16	15.6377	15.1913	15.9379	15.9379	16	1.31579	4.69072	0.16839	0.03335
17	16.6577	16.2323	16.9415	16.9415	17	1.30000	4.62000	0.16829	0.03335
18	17.6757	17.2696	17.9447	17.9447	18	1.28571	4.55340	0.16820	0.03335
19	18.6919	18.3037	18.9476	18.9476	19	1.27273	4.49057	0.16812	0.03335
20	19.7066	19.3349	19.9502	19.9502	20	1.26087	4.43119	0.16805	0.03334

The data represent the failure times for a particular model 84 Aircraft Windshield given Table 16.11 of Murthy et al. (2004). The data set is the strength data of glass of the aircraft window reported by Fuller et al (1994). The data are 0.040, 1.866, 2.385, 3.443, 0.301, 1.876, 2.481, 3.467, 0.309, 1.899, 2.610, 3.478, 0.557, 1.911, 2.625, 3.578, 0.943, 1.912, 2.632, 3.595, 1.070, 1.914, 2.646, 3.699, 1.124, 1.981, 2.661, 3.779,1.248, 2.010, 2.688, 3.924, 1.281, 2.038, 2.82,3, 4.035, 1.281, 2.085, 2.890, 4.121, 1.303, 2.089, 2.902, 4.167, 1.432, 2.097, 2.934, 4.240, 1.480, 2.135, 2.962, 4.255, 1.505, 2.154, 2.964, 4.278, 1.506, 2.190, 3.000, 4.305, 1.568, 2.194, 3.103, 4.376, 1.615, 2.223, 3.114, 4.449, 1.619, 2.224, 3.117, 4.485, 1.652, 2.229, 3.166, 4.570, 1.652, 2.300, 3.344, 4.602, 1.757, 2.324, 3.376, 4.663. The same data is studied by Ramos et al. (2013).

The choice of the best model for these data can be done with model selection criteria. In this section, the following goodness-of-fit tests are used, namely; Kolmogorov-Smirnov (KS), Cramer-von-Mises, (W) and Anderson-Darling (A). Additionally, four information criteria are considered including Akaike Information Criterion (AIC), Hannan-Quinn Information Criterion (HQIC), Bayesian Information Criterion (BIC), Consistent Akaike Information Criterion (CAIC) where these measures are defined as

$$\text{AIC} = -2MLL + 2t, \text{CAIC} = -2MLL + \frac{2t\,n}{n-t-1}, \text{BIC} = -2MLL + t\,log(n)$$

and

$$\text{HQIC} = 2Log\left\{Log(n)[t - 2MLL]\right\}$$

where t is the number of parameters and n is the sample size. The distribution with the lowest values of the AIC and BIC statistics is the best model for the data. Table 7 contains the results for the fitted models.

According to the results in Table 7, the LBMD distribution has the lowest values of these statistics, followed by MD, TSPD, JD and DD. Therefore, the suggested distribution is the best choice for the Aircraft Windshield data. The fitted pdfs and empirical cdfs plots of the five

Table 7: Model selection criteria for the Aircraft Windshield data.

Model	W	A	AIC	CAIC	BIC	HQIC	Statistic	P. value
LBMD	0.085377	0.818539	272.0738	272.2202	276.9591	274.0388	0.096644	0.405261
MD	0.084562	0.818225	276.7892	276.9356	281.6745	278.7543	0.130242	0.111835
DD	0.188065	1.540470	338.9284	339.0748	343.8137	340.8934	0.287137	1.64E-06
JD	0.148813	1.280334	293.5838	293.7301	298.4691	295.5488	0.182011	0.007165
TSPD	0.156153	1.329305	293.5074	293.6537	298.3927	295.4724	0.180809	0.007716

models are sketched in Figure 4. Therefore, we assert that the LBMD fitting successfully the empirical plots of the data set.

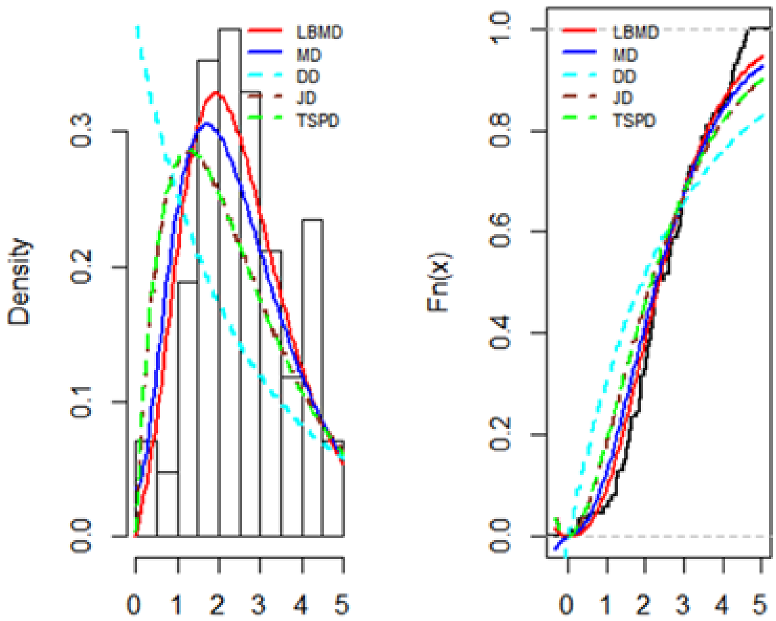


Figure 4: Estimated densities and cdfs of the models based on the real data.

5 Conclusions

In this paper, the LBMD is defined. Several structural properties of the LBMD are studied in details include the moments, moment generating function, reliability analysis, the coefficients of variation, skewness and kurtosis, distributions of order statistics and harmonic mean. Also, the maximum likelihood estimation is used to estimate the distribution parameters, and the Rényi entropy, stochastic ordering, mean and median deviations, and stress-strength are derived. Finally, an application to a real data set is considered to demonstrate the applicability of the suggested distribution in practical situations. The empirical results assert that the LBMD distribution could be considered as a competitive model and can offer better modeling ability than its counterparts considered in this study.

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